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COMPLETELY REGULAR ALGEBRAIC MONOIDS*

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By previous results of Putcha and the author, an irreducible algebraic monoid M is regular if and only if the Zariski closure $\overline{R(G)} \subseteq M$ is completely regular, where $R(G)$ is the solvable radical of G. Thus, the classification problem leads initially to extreme cases; reductive monoids and completely regular monoids with solvable unit groups.

In this paper we classify normal, completely regular (NCR) monoids with solvable unit group. It turns out that each NCR monoid M is determined by its unit group $G=TU$ and the closure Z of T in M. For the converse, we find the exact conditions on a diagram $\bar{T}=Z \hookleftarrow T \hookrightarrow G$ for which there exists an NCR monoid M with $Z=\bar{T} \subset M$.

Introduction

Let M be an irreducible algebraic monoid. By the results of [5,6], M is regular if and only if $\overline{R(G)} \subseteq M$ (Zariski closure) is completely regular. Here G is the (dense) unit group of M and $R(G)$ is the solvable radical of G. On the other hand, $M//R_u(G)$ (categorical quotient; see [6, Theorem 4.2]) is always regular since $G/R_u(G)$ is reductive. Thus, the classification problem leads initially to extreme cases; reductive monoids, and completely regular monoids. In [7], reductive (normal) monoids are classified numerically with toric data and root systems. In particular, these moduli are discrete.

In this paper, we classify normal, irreducible, completely regular (NCR) monoids in a similar spirit; we fix the unit group G and determine the exact nature of the toric data that distinguishes the different NCR monoids with unit group G. But there is also a converse. Given a toric datum of the above-mentioned type, we construct the unique monoid with that specification. A special case of our results was established by Mumford [3] (see [3, p. 182]). Another special case (Theorem 3.2 below) should be of interest in semigroup theory as an ideal model for the construction of a large class of orthodox monoids.

To state our main results, let $G=UT$ be a connected solvable group defined over

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the algebraically closed field k. Let M be an NCR monoid with unit group G and assume $0 \in \bar{T}$ (for simplicity). We have $\varrho = \mathrm{int} : T \to \mathrm{Aut}(U)$, which determines weights $\phi_T(U) \subseteq X(T)$. Let $X(\bar{T}) = \{\chi \in X(T) \mid \chi$ extends to $\bar{\chi} : \bar{T} \to k\}$.

Theorem 1. (a) $\phi_T(U) \subseteq X(\bar{T}) \cup -X(\bar{T})$.

(b) *Given a normal torus embedding* $T \subseteq \bar{T}$ *with* $0 \in \bar{T}$ *such that* $\phi_T(U) \subseteq X(\bar{T}) \cup -X(\bar{T})$, *there exists a unique monoid M which realizes these data.*

Theorem 2. *Let M be as above. For* $e \in E(\bar{T})$ *let* $U_+^e = \{u \in U \mid eu = e\}$ *and* $U_-^e = \{u \in U \mid ue = e\}$. *The following are equivalent*:

(a) *If* $e \in E(\bar{T})$, *then* $U_-^e U_+^e = U_+^e U_-^e$.

(b) *If* $e, f \in E(M)$, *then* $ef \in E(M)$.

1. Background

Let k be an algebraically closed field. An *algebraic monoid M*, is an affine, algebraic variety together with an associative morphism $m : M \times M \to M$ and a two-sided unit $1 \in M$ for m. $G = \{x \in M \mid x^{-1} \in M\}$ is an affine, open, algebraic subgroup of M. $E(M) = \{e \in M \mid e^2 = e\}$ is the set of *idempotents* of M and $E^1(M) = \{e \in E(M) \mid e \neq 1$ and e is maximal$\}$. A *toric monoid* (or *D*-monoid) Z is an irreducible, algebraic monoid such that $G \cong k^* \times \cdots \times k^*$. The normal toric monoids are precisely the affine, torus embeddings [2,3]. $X(Z) = \{\chi \in X(G) \mid \chi$ extends to $\bar{\chi} : Z \to k\}$ is the set of *characters* of Z.

A monoid M is *regular* if for each $x \in M$ there exists $a \in M$ such that $xax = x$. By the results of [4], an irreducible monoid is regular if and only if $M = GE(M) = E(M)G$. A monoid M is *completely regular* if $M = \bigcup_{e \in E(M)} H_e$, where H_e is the unit group of the monoid eMe (H_e is the *$\mathscr{H}$-class* of e). By the results of [5,6], it follows that an irreducible, algebraic monoid M is regular if and only if $\overline{R(G)}$ is completely regular, where $R(G) \lhd G$ is the solvable radical. In this paper we are interested in normal, completely regular, irreducible (NCR) monoids.

2. General results

Let U be a connected, unipotent, algebraic group defined over k, and let $\lambda : k^* \to \mathrm{Aut}(U)$ be a regular action by group automorphisms. Let $\mathscr{L}$ be the Lie algebra of U, identified canonically with the tangent space of U at 1. We assume $k^* \subseteq k$ fixed in perpetuity, and call it the *orientation* (since there are exactly two choices). By the results of [1], we have a direct sum decomposition

$$\mathscr{L} = \mathscr{L}_+ \oplus \mathscr{L}_0 \oplus \mathscr{L}_- \tag{1}$$

and corresponding *subgroups*

$$U_+ = \left\{ u \in U \,\middle|\, \lim_{t\to 0} \lambda_t(u) = 1 \right\},$$

$$U_0 = \{ u \in U \mid \lambda_t(u) = u \text{ for all } t \in k^* \},$$

$$U_- = \left\{ u \in U \,\middle|\, \lim_{t\to 0} \lambda_{t^{-1}}(u) = 1 \right\}.$$

Furthermore, U_0 normalizes U_+ and U_- (compute) and each group is connected. In particular, $U_+U_0 = U_\geq$ and $U_0U_- = U_\leq$ are connected subgroups of U. Notice that

$$\mu : U_+ \times U_0 \times U_- \to U \tag{2}$$

defined by $\mu(a,b,c) = abc$, is dominant, since the differential of μ (at 1) is addition, and $\mathscr{L}(U_+) = \mathscr{L}_+$, $\mathscr{L}(U_0) = \mathscr{L}_0$, $\mathscr{L}(U_-) = \mathscr{L}_-$. But $U_\geq$ is a subgroup of U, and so we have an action

$$\varphi : U_\geq \times U_0 \times U \to U$$

defined by $\varphi((a,b),c) = acb^{-1}$. By (2), φ has a dense orbit, namely $U_\geq U_- = U_+U_0U_-$. Thus, since unipotent group actions only have closed orbits,

$$U_+U_0U_- = U. \tag{3}$$

It also follows from the definitions that φ is bijective; but separable, from the differential computation, and hence an isomorphism by Zariski's main theorem (ZMT).

We now relate these results to the structure of completely regular monoids. We start with a special case.

Let M be an NCR monoid with unit group $G = k^*U \neq M$. Let $\overline{k^*} \subseteq M$ be the closure in M. By [8, Theorem 2.3], $\overline{k^*} \cong k$ (canonically) and we may assume that this agrees with the given orientation. Let U_+, U_0, U_- be as in (3) relative to the action $\lambda_t(u) = tut^{-1}$, of k^* on U. Let

$$M_\geq = \overline{k^*U_\geq}, \quad \text{and} \quad M_\leq = \overline{k^*U_\leq}.$$

These are closed NCR submonoids of M (the normality results from Theorem 2.5 below). Define

$$m_+ : U_\geq \times k \to M_\geq, \quad \text{and} \quad m_- : k \times U_\leq \to M_\leq,$$

the obvious multiplication maps.

2.1. Proposition. *m_+ and m_- are isomorphisms.*

Proof. If we do not assume $M_\geq$ is normal, we would prove (using the following arguments) that m_+ and m_- are *the* normalization morphisms. But then we would still obtain, from Theorem 2.5 below, that the multiplication map, $U_+ \times U_0 \times k \times U_- \to M$ is finite and birational, and so by ZMT, an isomorphism. So we may as well assume $M_\geq$ and $M_\leq$ are normal.

It suffices to show that m_+ is surjective, since any surjective birational morphism with normal target is an isomorphism (between affine varieties). But we can reduce that problem further. By the results of [5,6], and [4, Theorem 13], $M_{\geq} = G \cup GeG$, where $e = m_+(1,0)$. Thus, it suffices to show that image(m_+) is a submonoid.

By [3, p. 181, lemma], $\lambda^* : k^* \to \mathrm{Aut}(U_{\geq})$, $\lambda_t^*(u) = tut^{-1}$, extends to a morphism $\lambda : k \to \mathrm{End}(U_{\geq})$. Let $u^\alpha = \lambda_\alpha(u)$, for $\alpha \in k$ and $u \in U_{\geq}$. Define $(u,\alpha)\cdot(v,\beta) = (u \cdot v^\alpha, \alpha\beta)$. This turns $U_{\geq} \times k$ into an algebraic monoid such that m_+ is a morphism of algebraic monoids. □

2.2. Corollary. *Let* $e = m_+(1,0)$.

(a) $M_{\geq} = G \cup U_{\geq} e$;
(b) $U_{\geq} \xrightarrow{\cong} U_{\geq} e$;
(c) $eU_{\geq} = eU_0$;
(d) $U_0 \xrightarrow{\cong} U_0 e = eU_0$;
(e) $U_- = \{u \in U \mid ue = e\}$;
(f) $U_+ = \{u \in U \mid eu = e\}$.

Similar results hold for $M_{\leq}$.

Proof. Compute directly with the isomorphism of 2.1 and the multiplication table $(u,\alpha)(v,\beta) = (uv^\alpha, \alpha\beta)$ remembering that $u^0 \in U_0$ for all $u \in U_{\geq}$. □

Remark. Let M_1 be any NCR monoid and let $k^* \subseteq T$ be such that $e \in \overline{k^*}$, $e^2 = e \neq 1$. Then take $M = \overline{Uk^*} \subseteq M_1$. If follows easily that 2.2 holds for the monoid M and the idempotent e.

So now we let M be any NCR monoid such that $0 \in \bar{T} \subseteq M$. (This is the zero of $\bar{T}$ *not* of M). If $e \in E^1(\bar{T})$, then there is a *unique* $k^* \subseteq T$ such that $e \in \overline{k^*}$. This defines a homomorphism

$$\gamma_e : X(T) \to X(k^*) \cong Z$$

dual to the inclusion $k^* \subseteq T$.

Now $G = TU$, and by (1) above, $\mathscr{L} = \mathscr{L}(U) = \mathscr{L}_+ \oplus \mathscr{L}_0 \oplus \mathscr{L}_-$ relative to the $k^* \subseteq T$ of $e \in E^1(\bar{T})$. Let

$$\phi_T(U) = \{\alpha \in X(T) \mid \mathscr{L}_\alpha \neq (0)\},$$

where $\mathscr{L} = \bigoplus_\alpha \mathscr{L}_\alpha$, via the adjoint action.

2.3. Theorem. *Let* $0 \neq \alpha \in \phi_T(U)$ *and let* $e, f \in E^1(\bar{T})$. *Then*

$$\gamma_e(\alpha)\,\gamma_f(\alpha) \geq 0.$$

Proof. Assume not. Then without loss of generality, $\gamma_e(\alpha) < 0$ and $\gamma_f(\alpha) > 0$. So let

$$U_- = \{u \in U \mid ue = e\}$$

and

$$U_+ = \{u \in U \mid fu = f\}.$$

By (2) above and 2.2(e), (f),

$$\mathscr{L}_\alpha \subseteq \mathscr{L}(U_-) \cap \mathscr{L}(U_+).$$

Let $L \subsetneq \mathscr{L}_\alpha$ be a one-dimensional subspace. By [1, Theorem 2.1], there exists a closed subvariety $V \subseteq U$, such that $1 \in V$ is a nonsingular point, $T_1(V) = L$ and $tVt^{-1} = V$ for all $t \in T$. So if $\lambda_e : k^* \to \bar{T}$ and $\lambda_f : k^* \to \bar{T}$ are the one parameter subgroups of e and f respectively, then both $\overline{\text{int}} \circ \lambda_e$ and $\text{int} \circ \lambda_f$ extend from k^* to k, where $\text{int}(t)(u) = tut^{-1}$ and $\overline{\text{int}}(t)(u) = t^{-1}ut$. (This follows from the weight argument on [3, p. 181], since $\gamma_e(\alpha) < 0$ and $\gamma_f(\alpha) > 0$.)

Hence, by the description of U_+ and U_- just following (1),

$$V \subseteq U_+ \cap U_-.$$

This contradicts 2.2(d) (applied to ef), since by definition, $efV = Vef = \{ef\}$. $\square$

2.4. Corollary. $\Phi_T(U) \subsetneq X(\bar{T}) \cup -X(\bar{T})$.

Proof. By the results of [2], $X(\bar{T}) = \{\chi \in X(T) \mid \gamma_e(\chi) \geq 0$ for all $e \in E^1(\bar{T})\}$. But by Theorem 2.3, $\gamma_e(\alpha)$ and $\gamma_f(\alpha)$ have the same sign for $e, f \in E^1(\bar{T})$ and $\alpha \in \phi_T(U)$. $\square$

Now that we have most of the technical details in place, we fix T, and $e_0 \in \bar{T}$ the minimal idempotent. By our assumptions, $e_0 T = Te_0 = \{e_0\}$, and by the remark following 2.2,

$$\begin{aligned} U_+ &= \{u \in U \mid e_0 u = e_0\}, \\ U_- &= \{u \in U \mid ue_0 = e_0\}, \\ U_0 &= \{u \in U \mid ue_0 = e_0 u\} \end{aligned}$$

satisfy the conclusions of Corollary 2.2. Furthermore, the multiplication map $U_+ \times U_0 \times U_- \to U$ is bijective. By the standard weight argument [3, p. 181],

$$T \times U_+ \to U_+, \qquad (t, u) \mapsto tut^{-1},$$

extends to an action a_+ of $\bar{T}$ on U_+ (by algebraic group endomorphisms). Similarly, we have an action

$$a_- : \bar{T} \times U_- \to U_-$$

extending the action $(t, u) \mapsto t^{-1}ut$ of T on U_-. We denote

$$a_+(x, u) \text{ by } u^x, \quad \text{and} \quad a_-(y, u) \text{ by } u^{\bar{y}}.$$

2.5. Theorem. *Let M be an NCR monoid with $0 \in \bar{T}$ and let $M_0 = \overline{TU_0} \cong \bar{T} \times U_0 \subseteq M$. Define*

$$m : U_+ \times M_0 \times U_- \to M$$

by $m(x, y, z) = xyz$. Then M is an isomorphism of algebraic varieties.

Proof. Just as in the proof of 2.1, it suffices to show that the image of m is a submonoid. For that, we define a monoid structure on $U_+ \times M_0 \times U_-$ so that m is a monoid homomorphism. By (3) we have $U_- U_+ \subseteq U = U_+ U_0 U_- \cong U_+ \times U_0 \times U_-$. This determines $\zeta_+ : U_- \times U_+ \to U_+$, $\zeta_0 : U_- \times U_+ \to U_0$ and $\zeta_- : U_- \times U_+ \to U_-$; and these are all morphisms of varieties. So $uv = \zeta_+(u,v)\zeta_0(u,v)\zeta_-(u,v)$, for $u \in U_-$, $v \in U_+$. Define a product on $U_+ \times M_0 \times U_-$ as follows:

$$(u,x,v)(a,y,b) = (u\zeta_+(v,a)^x, x\zeta_0(v,a)\,y, \zeta_-(v,a)^{\bar{y}}b) \tag{4}$$

Let $G_0 = TU_0 = U_0 T$. It is easily checked that if uxv, $ayb \in G = U_+ G_0 U_-$, then

$$(uxv)(ayb) = u\zeta_+(v,a)^x x\zeta_0(v,a)\,y\zeta_-(v,a)^{\bar{y}}b.$$

Hence, the product in (4) agrees with the group law on $G \cong U_+ \times G_0 \times U_-$ and so it is associative on $M = U_+ \times M_0 \times U_-$, because it is a morphism of varieties and $G \subseteq M$ is dense. m is a morphism of algebraic monoids, again by the density argument. □

2.6. Converse. *Let U be a connected unipotent algebraic group and suppose we are given the following data*:

(a) *A torus action $\varrho : T \to \mathrm{Aut}(U)$, by algebraic group automorphisms.*

(b) *A normal torus embedding $T \to \bar{T}$ such that $0 \in \bar{T}$ and $\phi_T(U) \subseteq X(\bar{T}) \cup -X(\bar{T})$.*

Then there exists a unique structure of an NCR monoid on $U_+ \times \bar{T} \times C_U(T) \times U_-$ extending the group law on $U_+ \times T \times C_U(T) \times U_- \cong U \ltimes_\varrho T$.

Proof. This is straightforward. $M_0 = \bar{T} \times C_U(T)$ is an algebraic monoid with the product structure. So we can use the formula (4) to define the monoid law on $M = U_+ \times M_0 \times U_-$. □

3. Some special cases

In this section we distinguish certain special cases of NCR monoids.

3.1. Corollary. *Let $M = U_+ M_0 U_-$ be an NCR monoid. The following are equivalent*:

(a) $U_- = \{1\}$.

(b) $\phi_T(U) \subseteq X(\bar{T})$.

(c) *M is considered on page* 182 *of* [3].

(d) $U = U_+ U_0$.

Proof. Obvious. □

3.2. Theorem. *Let M be an NCR monoid and let $U_-^e = \{u \in U \mid ue = e\}$ and $U_+^e = \{u \in U \mid eu = e\}$. The following are equivalent*:

(a) *If* $e \in E(\bar{T})$, *then* $U_-^e U_+^e = U_+^e U_-^e$.
(b) *If* $e, f \in E(M)$, *then* $ef \in E(M)$.
(c) M *is orthodox.*

Proof. (b) and (c) are equivalent by the definition. Let $e, f \in E(\bar{T})$ and define $\eta_+ : U_-^e \times U_+^f \to U_+^{ef}$, $\eta_0 : U_-^e \times U_+^f \to U_0^{ef}$ and $\eta_- : U_-^e \times U_+^e \to U_-^{ef}$ by the inclusion $U_-^e U_+^f \subseteq U_+^{ef} U_0^{ef} U_-^{ef}$ (see the proof of Theorem 2.5). Thus, if $u \in U_-^e$ and $v \in U_+^f$, we have

$$uv = \eta_+(u, v)\eta_0(u, v)\eta_-(u, v).$$

Now, by 2.1 and 2.5, any $x \in M$ can be written uniquely as $z = (u, x, v)$, where $x = tae$ with $t \in T$, $e \in E(\bar{T})$ and $a \in C_U(e)$, and $u \in U_+^e$ and $v \in U_-^e$. u, x and v are called the *e-coordinates* of z. Further, it is easy to check that z is idempotent if and only if $te = e$ and $u = 1$. So let (u, e, v) and (a, f, b) be idempotents. (So these elements are in different coordinate systems.) One computes and then obtains,

$$(u, e, v)(a, f, b) = (u\eta_+^e, e\eta_0 f, \eta_-^{\bar{f}} b) = \omega$$

where $\eta_+ = \eta_+(v, a)$, $\eta_0 = \eta_0(v, a)$ and $\eta_- = \eta_-(v, a)$. The right-hand side is in ef-coordinates since $U_0^{ef} \subseteq U_0^e \cap U_0^f$, $U_+^e \subseteq U_+^{ef}$ and $U_-^f \subseteq U_-^{ef}$. But ω is idempotent if and only if $e\eta_0 f = ef$. By 2.2(d), this happens if and only if $\eta_0 = 1$. Since $v \in U_-^e$ and $a \in U_+^f$ can be arbitrarily so, we see that ω is always idempotent if and only if $\eta_0 : U_-^e \times U_+^f \to U_0^{ef}$ is trivial. Thus, (b) is equivalent to the condition

$$U_-^e U_+^f \subseteq U_+^{ef} U_-^{ef}.$$

But this is equivalent to (a) since $U_-^e U_+^f \subseteq U_-^{ef} U_+^{ef}$. $\square$

4. Concluding remarks

4.1. 2.5 and 2.6 effectively classify all normal, affine embeddings X of a solvable group $G = TU$ with the two-sided action and the property $G\bar{T}G = X$. Unlike the special case of 3.1, there is an intriguing array of embeddings $T \to \bar{T}$ with $\phi_T(U) \subseteq X(\bar{T}) \cup -X(\bar{T})$.

4.2. In the proof of 2.5 we defined the product law on $U_+ \times M_0 \times U_-$ in terms of the $\bar{T}$-action on U, and the functions ζ_+, ζ_- and ζ_0 (see (4)). The associativity was then proved by a topological argument. It is likely that a semigroup-theoretic axiomatization of 2.5, 2.6 and 3.2 would be possible directly. This should reflect a clearer light on the monoids involved here.

4.3. In [7], reductive regular monoids are classified numerically, while in [5] it is shown (taking into account [6]) that an irreducible monoid is regular if and only if $\overline{R(G)}$ is completely regular. Thus, by the results of this paper we have an essentially complete theory for the two extremes involved in the general case. However, the general classification problem is still open.

4.4. It follows directly from 3.2(a) that if $G = TU$ and U is abelian, then M is orthodox.

4.5. It follows from 2.5 and the main results of [2] that any NCR monoid is a Cohen–Macaulay algebraic variety.

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